These notes were compiled for a reading group presentation, to offer additional material, practice, and possible motivations for research. These notes haven't been put through the usual scrutiny, so please suggest corrections if you see any errors.

The main references used are [Spe98] and [Leh99].

Note that the discussion of operator-valued FPT in [Spe98] emphasises heavily on the amalgamated free product of algebras, in hopes to build the moment map by patching together the respective moment maps defined within the free sub-algebras. We do not particularly need such a construction.

1 Recap and motivation

We are now (more than) halfway through the reading group this semester. At this point, let's look back at the progress made towards the main goal for the reading group, and review some key takeaways we have seen so far.

1.1 Recap

Until now, we have been considering (scalar-valued) free probability, mainly through a combinatorial lens. We have defined, and characterised, free independence in terms of the free moments and also in terms of the free cumulants. To that end, we have also seen how the free cumulants and the free moments uniquely determine each other, via some specific convolutions.

We then considered free additive and multiplicative convolutions of free random elements.

Some useful tools we developed have been the study of the $\mathrm{NC}(n)$ lattice, the ζ and μ functions defined on $\mathrm{NC}(n) \times \mathrm{NC}(n)$ (and later expressed as formal power series), the concept of the resolvent, and certain key transforms of free random elements.

1.1.1 The full Fock space

In this part, we aim to review some important notions about the full Fock space. Let $\mathcal{H}$ be some Hilbert space; we restrict ourselves to the setting where $m = \dim \mathcal{H} < \infty$ for simplicity. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis for $\mathcal{H}$. The full Fock space on $\mathcal{H}$ is the Hilbert space given by

$$\mathcal{T}(\mathcal{H}) := \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}.$$

Here, $\mathbb{C}\Omega$ can be thought of as the stand-in for $\mathcal{H}^{\otimes 0} \cong \mathbb{C}$, Ω is called the *vacuum vector*. Given some $v \in \mathcal{H}$, we define the corresponding *creation operator* $l_v \in \mathcal{B}(\mathcal{T}(\mathcal{H})) =: \mathcal{A}$ as a bounded operator such that

$$\begin{aligned} l_v &: \Omega \mapsto v \\ l_v &: e_{i_1} \otimes \ldots e_{i_n} \mapsto v \otimes e_{i_1} \otimes \ldots e_{i_n} \end{aligned}$$

It's convenient to use the shorthand $l_i := l_{e_i}$. Recall also that the adjoint l_v^* is the *annihilation operator*:

$$\begin{aligned} l_v^* &: \Omega \mapsto 0 \\ l_v^* &: e_{i_1} \mapsto \Omega \\ l_v^* &: e_{i_1} \otimes \ldots e_{i_n} \mapsto \langle e_{i_1}, v \rangle \, e_{i_2} \otimes \ldots e_{i_n} \end{aligned}$$

We defined the vacuum state expectation as $\tau_{\mathcal{H}}(T) := \langle T\Omega, \Omega \rangle$ for all $T \in \mathcal{A}$. Then, the pair $(\mathcal{A}, \tau_{\mathcal{A}})$ is a C^*-probability space, and $\{l_i + l_i^*\}_{i \in [m]}$ forms a semicircular system in $(\mathcal{A}, \tau_{\mathcal{H}})$.

1.2 Motivation: the X_{free} model

As in [BBvH23], we seek to study asymptotic properties of the noncommutative random matrix model, defined as follows:

$$X^N = A_0 \otimes \mathrm{Id}_N + \sum_{i=1}^n A_i \otimes G_i^N.$$

Here, $A_0, A_1, \ldots, A_n \in M_d(\mathbb{C})_{\text{sa}}$, $A_0 \succeq 0$, and the G_i^N are standard Wigner matrices where each matrix entry is sampled iid from a centred Gaussian distribution with variance $1/N$, i.e. $G_i^N = (g_i^{jk})_{j,k=1}^N \sim_{iid} \mathcal{N}(0, \frac{1}{\sqrt{N}})$.

It follows from the free central limit theorem that, in the limit $N \to \infty$, the X^N converge in distribution to

$$X_{\text{free}} := A_0 \otimes \mathrm{id} + \sum_{i=1}^n A_i \otimes s_i, \tag{1}$$

where $(s_1, \ldots, s_n)$ is a free semicircular family in some $\mathcal{A}$, as operators on some full Fock space, and id denotes the identity map in $\mathcal{A}$. We can then interpret $X_{\text{free}} \in (M_d(\mathbb{C}) \otimes \mathcal{A}, \varphi)$, where $\varphi := \mathrm{tr} \otimes \tau_{\mathcal{H}}$.

It is now of our interest to compute the free moments of this new X_{free} object, so we can study asymptotic properties of the X^N. With some effort (apparently from [NS06]; I don't know the details), we get

$$\varphi(X_{\text{free}}^{2p}) = \sum_{\pi \in \mathrm{NC}_2(2p)} \sum_{(i_1, \ldots, i_{2p}) \sim \pi} \mathrm{tr}[A_{i_1} \ldots A_{i_{2p}}],$$

where $(i_1, \ldots, i_{2p}) \sim \pi$ denotes $i_j = i_k$ if $j \sim_\pi k$ for all $1 \leq j < k \leq n$. Note that this formula for the free moments doesn't have a multiplicative structure with respect to each fixed $\pi \in \mathrm{NC}(n)$, and hence, cannot be factored over the $\mathrm{NC}(n)$ lattice as we have been doing until now. As a result, we cannot naturally use these free moments to determine the corresponding cumulants, which in turn, leaves us unable to study any convolutions, and so on.

The hope now is that, perhaps, choosing φ to be a $\mathbb{C}$-valued functional omits too much structure. We then consider preserving some of the matrix structure of X_{free} and developing a new, but similar, line of FPT accordingly.

2 Operator-valued FPT

The main takeaway from this section should be that operator-valued FPT replaces almost every instance of maps to $\mathbb{C}$ with an analogous map to B, where B is some arbitrary unital C^*-algebra. When in doubt, or in need of additional examples, it may be helpful to consider the case $B = \mathbb{C}$ (which retrieves the scalar-valued FPT) or $B = M_m(\mathbb{C})$ (for any fixed m).

This section roughly follows the same order in which we approached scalar-valued FPT, and is organised as follows. We begin with some motivation and basic definitions, before defining what shall be our operator-valued moment and cumulant maps, defined with respect to the $\mathrm{NC}(n)$ lattices as any good free probability theorist should do. Next, we prove that these operator-valued moments and cumulants are indeed *multiplicative* (to be defined soon), and hence satisfy our problem with the $\varphi(X_{\text{free}}^{2p})$ moments earlier. Immediately capitalising on this, we start defining and operator-valued freeness and $*$-distributions, which we shall use to end the section with introducing all the key transforms.

2.1 Motivation

We derive inspiration from the conditional expectation map in classical probability theory. Indeed, suppose $X_1, \ldots, X_t$ are some non-commutative (classical) random variables, e.g., we can consider them to be matrices. Then, upon conditioning on X_1, we note the following:

- $\mathbb{E}[X_1 | X_1] = X_1$;

- $\mathbb{E}[X_1 X_2 \dots X_t | X_1] = X_1 \mathbb{E}[X_2 \dots X_t | X_1] \neq \mathbb{E}[X_2 \dots X_t | X_1] X_1$;
- For any polynomial p, $\mathbb{E}[X_1 p(X_2, \dots, X_t) X_1 | X_1] = X_1 \mathbb{E}[p(X_2, \dots, X_t) | X_1] X_1$.

With these observations in hand, we seek to define a *free* analogue of conditional expectation which satisfies equivalent statements in FPT.

2.2 Definitions

In what follows, suppose that B is always some arbitrary finite-dimensional unital C^*-algebra with unit $1_B \in B$.

Definition 2.1 (Algebra over B)**.** *A $*$-algebra A is said to be an algebra over B when $B \subseteq A$ is a sub-algebra of A.*

In particular, when given an algebra over B, note that B will remain closed under multiplication and adjoint. We shall usually only consider the case where A itself is unital C^*-algebra, with unit $1_A \in A$.

Example 2.2. *Consider $B \cong \mathbb{C}$ and $A = M_2(\mathbb{C})$, where B consists of all matrices where all entries, except the top-left entry, are 0, i.e., $B = \left\{ \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} : b \in \mathbb{C} \right\}$. Clearly, both A and B are unital C^*-algebras and B is a sub-algebra of A, so A is an algebra over B. The respective units are $1_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $1_A = \mathrm{Id}_2$.*

Warning! During the presentation, I said that $1_B = 1_A$. This is clearly false, as shown in the above example!

Note that, for any algebra A over B, we have an inclusion map $\iota : B \hookrightarrow A$. However, for any $b \in B$, we shall overload notation to also write $b \in A$ when referring to the element $\iota(b) \in A$.

With the following proposition as a sanity check, we remark that when $B \cong \mathbb{C}$ as a C^*-algebra, we retrieve the scalar-valued setting.

Proposition 2.3. *Any B-functional φ is unital, i.e., $\varphi(1_A) = 1_B$.*

Proof. Let $b \in B$ be arbitrary. Then, $b = \varphi(b) = \varphi(b 1_A) = b\varphi(1_A)$. Since this holds for all $b \in B$, the only possibility is $\varphi(1_A) = 1_B$. ■

We make one additional definition, as follows.

Definition 2.4 (B-functional)**.** *A B-functional on A is a linear map $\varphi : A \to B$ such that $\varphi(b) = b$, for all $b \in B$, and $\varphi(b_1 a b_2) = b_1 \varphi(a) b_2$, for all $b_1, b_2 \in b$ and $a \in A$.*

We say that such a map φ is a *free conditional expectation* conditioned on B, or a B-valued expectation. Indeed, we verify the following, satisfied it meets our picky demands we set forth:

- $\varphi(b) = b$; and $\varphi(b a_1 \dots a_t b) = b \varphi(a_1 \dots a_t) b$;
- $\varphi(b p(b, a_1, \dots, t) b) = b \varphi(p(b, a_1, \dots, a_t)) b$

Accordingly, we can now define a B-valued $*$-probability space.

Definition 2.5 (B-valued $*$-probability space)**.** *This refers to a pair (A, φ) where A is a $*$-algebra over B and $\varphi : A \to B$ is a positive B-functional.*

Corollary 2.5.1. *When $B \cong \mathbb{C}$ as a C^*-algebra (equiv., when $\dim B = 1$), we retrieve scalar-valued FPT.*

Here, we say $\varphi : A \to B$ is positive in the conventional sense, i.e., for all $a \in A$, $\varphi(a^* a) = b^* b$ for some $b \in B$.

Consider the following examples for such a B-valued probability space.

Example 2.6. *Let (A, τ) be a C^*-probability space. Consider $M_d(A)$, the set of all $d \times d$ matrices with entries in A, which is itself a C^*-algebra under the standard operations. Note that $B \cong M_d(\mathbb{C})$ is a C^*-subalgebra of $M_d(A)$. Upon verifying that the map $\varphi : M_d(A) \to M_d(\mathbb{C})$ given by $\varphi := \mathrm{Id}_d \otimes \tau$ is a B-functional, it follows that $(M_d(A), \mathrm{Id}_d \otimes \tau)$ is a B-valued C^*-probability space.*

Example 2.7. *In particular, if A is a matrix algebra $M_n(\mathbb{C})$ and $\tau = \mathrm{tr}$ is the normalised trace on A, then the above construction yields the conditional expectation given by the partial trace $\varphi = \mathrm{Id}_d \otimes \mathrm{tr}$.*

And, perhaps most relevant for our purposes, we momentarily revisit our good friend, the X_{free} model.

Example 2.8 (The $M_d(\mathbb{C}) \otimes \mathcal{A}$ model)**.** *Note that $M_d(\mathbb{C}) \cong M_d(\mathbb{C}) \otimes \{1_{\mathcal{A}}\}$ is a C^*-subalgebra of $\mathcal{A}_d := M_d(\mathbb{C}) \otimes \mathcal{A}$. Moreover, the map $\varphi := \mathrm{Id}_d \otimes \tau_{\mathcal{A}}$ gives a $M_d(\mathbb{C})$-functional. Thus, the space $(\mathcal{A}_d, \varphi)$ is the operator-valued model for X_{free} that we are interested in.*

An interesting observation here is that you can compose together this notion of B-valued conditional expectations, in the following sense.

Remark 2.9. *Let (B, ϕ) be a C-valued C^*-probability space, where C is another unital C^*-algebra, and let (A, φ) be a B-valued $*$-probability space. Then, $(A, \phi \circ \varphi)$ is a C-valued $*$-probability space.*

This is analogous to the tower property of classical conditional expectations; e.g., if X_1, X_2, X_3 are classical random variables, then

$$\mathbb{E}[X_3|X_1] = \mathbb{E}[\mathbb{E}[X_3|X_2]|X_1].$$

I don't know if it is useful in FPT or random matrix theory, but it seems like a result that is often unnoticed.

2.3 Moments and cumulants

The goal here is simple: we are interested in computing $\varphi(X_{\mathrm{free}}^{2p})$ in a structure-preserving way. To do so, we build up the moment and cumulant formulae in this generalised setting.

We seek to define our moment and cumulant maps as B-functionals on $\bigsqcup_{n\in\mathbb{N}} \mathrm{NC}(n) \times A^{\otimes n}$ (up to a minor inconsistency, explained soon). By explicitly considering their action on non-crossing partitions, just like for scalar-valued FPT, we are able to guarantee that this definition admits a factoring over the $\mathrm{NC}(n)$ lattices. Recall that, when we mention a *factoring*, we mean to say that the functional respects the bracketing provided by any $\pi \in \mathrm{NC}(n)$.

However, to do so, we must first consider what exactly it means for a B-functional to be *multiplicative*, and thus admit such a factoring. Section 2.3.1 serves this purpose, but can be skipped if you wish to proceed directly to defining moments and cumulants.

2.3.1 Multiplicative functionals

Before we proceed any further, note that the conditional expectation imposes a strange requirement, i.e. $\varphi(ba) = b\varphi(a)$ and $\varphi(ab) = \varphi(a)b$. Clearly, then, when we consider a tensor product structure, we should have the flexibility to *"move around"* products by b. For this purpose, we define the tensor product with respect to $\otimes_B$ in Section 4.1, but it can be waived away as just some extra notation.

The goal now is to characterise all B-functionals $\hat{f} : \bigsqcup_{n\in\mathbb{N}} \mathrm{NC}(n) \times A^{\otimes_B n}$ such that $\hat{f}(\pi) : A^{\otimes_B n}$ respects the bracketing of π. To that end, we provide the following recursive definition.

Definition 2.10 (Multiplicative B-functional)**.** *Let $f^{(n)} : A^{\otimes_B n} \to B$ be arbitrary B-functionals, for $n \in \mathbb{N}$. We say $\hat{f} : \bigsqcup_{n\in\mathbb{N}} \mathrm{NC}(n) \times A^{\otimes_B n}$ is multiplicative, using $f^{(n)}$, when it obeys:*

1. *(base case.) $\hat{f}(\emptyset)[b] = b$;*

2. *(recursive case.) Consider $\hat{f}(\pi)[a_1 \otimes_B \ldots \otimes_B a_n]$, where $\pi \in \mathrm{NC}(n)$ and $n > 0$. Let $V = [k, \ell]$ denote the leftmost interval in π. Then,*

$$\hat{f}(\pi)[a_1 \otimes_B \ldots \otimes_B a_n] = \hat{f}(\pi \setminus V)[a_1 \otimes_B \ldots \otimes_B a_{k-1} \otimes_B f^{(\ell-k+1)}(a_k \otimes_B \ldots \otimes_B a_\ell)a_{\ell+1} \otimes_B \ldots a_n]$$

Consider the following computation, which illustrates why we call such a functional *multiplicative*.

Example 2.11. *Let $\pi = \{\{1,4\},\{2,3\}\} \in \mathrm{NC}(4)$; then its leftmost interval is $V = [2,3]$, and $\pi \setminus V = \{1,4\} \in \mathrm{NC}(2)$. Consider an arbitrary family $f^{(n)}$ of B-functionals. Then, for $a \in A$:*

$$\begin{aligned}\hat{f}[\pi](a \otimes_B a \otimes_B a \otimes_B a) &= \hat{f}[\pi \setminus V](a \otimes_B f^{(2)}(a \otimes_B a)a) \\ &= \hat{f}[\emptyset](f^{(2)}(a \otimes_B f^{(2)}(a \otimes_B a)a)) \\ &= f^{(2)}(a \otimes_B f^{(2)}(a \otimes_B a)a)\end{aligned}$$

In particular, as shown above $\hat{f}[\pi]$ respects the bracketing of π, in this example.

We claim (without proof) that such a bracketing always occurs with our definition for $\hat{f}(\pi) : A^{\otimes_B n} \to B$, for any $\pi \in \mathrm{NC}(n)$, and any $n \in \mathbb{N}$, which justifies the *multiplicative* naming.

2.3.2 Back to moments and cumulants

Definition 2.12 (Operator-valued moments)**.** *The multiplicative moment functional $\hat{\varphi}$ is determined by $\varphi^{(n)}$, where (for $n \geq 2$):*

$$\varphi^{(1)}(a) := \varphi(a) \tag{2}$$

$$\varphi^{(n)}(a_1 \otimes_B \ldots \otimes_B a_n) := \varphi(a_1 \ldots a_n) \tag{3}$$

At this point, recall that we characterised scalar-valued free cumulants using the ζ function; we shall repeat the same definition for B-valued cumulants also. As a reminder, we previously defined the function $\zeta : \sqcup_n \mathrm{NC}(n) \times \mathrm{NC}(n) \to \mathbb{C}$ as the indicator function for intervals, i.e.,

$$\zeta(\sigma, \pi) = \begin{cases} 1, & \sigma \leq \pi \\ 0, & \text{else} \end{cases}$$

Recall that ζ has a convolutive inverse given by the Möbius function $\mu : \sqcup_n \mathrm{NC}(n) \times \mathrm{NC}(n) \to \mathbb{C}$.

Definition 2.13 (Operator-valued cumulants)**.** *Given B-valued moments $\hat{\phi}$, we define the corresponding cumulants as $\hat{\kappa}$ given by $\kappa^{(n)} : A^{\otimes_B n} \to B$ where*

$$\hat{\kappa} := \hat{\varphi} \star \mu.$$

Here, for any $\pi \in \mathrm{NC}(n)$, the convolution is given by

$$(\hat{\varphi} \star \mu)(\pi) := \sum_{\sigma \leq \pi} \hat{\phi}(\sigma)\mu(\sigma, \pi) \tag{4}$$

The $\star$-convolution as given in Equation (4) can be generalised for any $\hat{f}$ and any function $\eta : \sqcup_n \mathrm{NC}(n) \times \mathrm{NC}(n) \to \mathbb{C}$. Moreover, this yields the first half of our operator-valued moment-cumulant formulae. The other half is obtained via the following corollary.

Corollary 2.13.1. *Given B-valued cumulants $\hat{\kappa}$, the corresponding B-valued moments are $\hat{\varphi} = \hat{\kappa} \star \zeta$.*

Proof. Recall that $\mu * \zeta = \mathrm{id}$, and further observe that the $\star$-convolution is *"associative"* in the sense that $(\hat{f} \star \nu_1) \star \nu_2 = \hat{f} \star (\nu_1 * \nu_2)$. ∎

By definition, then, the B-valued moments and cumulants are multiplicative, as desired. Moreover, the same moment-cumulant formulae from scalar-valued FPT still hold! We shall soon see that this is a recurring theme in operator-valued FPT, and that most of the same characterisations and functional equations from scalar-valued FPT carry over.

For instance, the same functional equation between the moments and cumulants is still true, as follows.

Theorem 2.14. *(Functional relation for moments and cumulants) Given B-valued moments $\hat{\varphi}$ and cumulants $\hat{\kappa}$, define the corresponding formal power series M and C as*

$$M(a) = 1 + \sum_{n=1}^{\infty} \varphi^{(n)}(a^{\otimes n})$$
$$C(a) = 1 + \sum_{n=1}^{\infty} \kappa^{(n)}(a^{\otimes n}).$$

Then, we have the functional relation $C[aM(a)] = M(a)$.

We omit the proof here, since it follows by direct computation, and is very similar to the proof of the functional relation for scalar-valued moments and cumulants.

With all these tools regarding B-valued moments and cumulants in hand, we are now ready to define what it means for two B-valued random variables $a, a' \in A$ to be freely independent.

2.4 Freeness

The following serves as both a definition and a characterisation theorem for B-valued free independence. As such, while it is labelled as a definition here, there are several nontrivial steps involved in justifying the result, which we shamelessly sweep under the rug.

Definition 2.15 (Free independence). *Given unital C^*-sub-algebras $(A_i)_{i\in\Lambda} \subseteq A$ such that each A_i is also a C^*-algebra over B, we say the $(A_i)_\Lambda$ are free when, for al $n \in \mathbb{N}$, $\varphi(a_1 \dots a_n) = 0 \in B$ whenever:*

1. *$a_j \in A_{i_j}$ for all $1 \leq j \leq n$;*
2. *$\varphi(a_j) = 0 \in B$ for all $1 \leq j \leq n$; and*
3. *all neighbouring elements are from different sub-algebras.*

Equivalently, the $(A_i)_\Lambda$ are free when all their mixed B-valued cumulants vanish. In other words, for all $n \in \mathbb{N}$, $\kappa^{(n)}(a_1 \otimes_B \dots \otimes_B a_n) = 0$ if there exist $1 \leq j < j' \leq n$ such that $A_{i_j} \neq A_{i_{j'}}$.

The usual route would be to now consider what the generalisation of a $*$-distribution looks like in the operator-valued setting. However, as an experiment, it might be less confusing to instead dive straight into considering the operator-valued transforms instead. This is because the results on matrix concentration and computing norms only ever use the definitions of the transforms themselves, instead of how they are defined with respect to (operator-valued) distributions, which takes some work to construct.

2.5 Transforms

We introduce all relevant transforms as formal power series objects, or functions thereof. For starters, recall that we defined the Cauchy transform $G_a(z)$ in scalar-valued FPT as $G_a(z) = \frac{1}{z}\sum_{n=0}^{\infty} \frac{\varphi(a^n)}{z^n}$, where $z \in \mathbb{C}_+$. Then, note that:

$$G_a(z) = \frac{1}{z}\sum_{n=0}^{\infty} z^{-n}\varphi(a^n) = z^{-1}\sum_{n=0}^{\infty}\varphi((z^{-1}a)^n) = z^{-1}\varphi(\textstyle\sum_{n=0}^{\infty}(z^{-1}a)^n) = z^{-1}\varphi((1_A - z^{-1}a)^{-1}),$$

where the last result follows from a C^*-algebraic analogue of the convergence of an infinite geometric series. Thus, we could express the Cauchy transform as the expectation of the resolvent:

$$G_a(z) = \varphi(z^{-1}(1_A - z^{-1}a)^{-1}) = \varphi((z1_A - a)^{-1})).$$

We choose to co-opt this definition, since it yields a natural generalisation by replacing z with some operator, as follows.

Definition 2.16 (Operator-valued Cauchy transform). *Given a B-valued $*$-probability space (A, φ), and $a \in A$, we define $G_a(b) := \varphi((b - a)^{-1}) \in B$.*

In other words, we have defined the operator-valued Cauchy transform as a map $G_a : B \to B$. However, (as we shall soon see) it will be more practical to restrict it to $G_a : \mathbb{H}^+(B) \to G_a(\mathbb{H}^+(B))$, where $\mathbb{H}^+(B)$ denotes the positive cone of B (analogous to the upper half-plane $\mathbb{C}_+$ for $\mathbb{C}$).

Continuing with co-opting our definitions from scalar-valued FPT, we now define the moment-transform $\tilde{G}_a(b)$, which corresponds to the previous moment transform $M_a(z)$.

Definition 2.17 (Operator-valued moment transform)**.** *For $a \in A$, we define $\tilde{G}_a(b) := \sum_{n=0}^{\infty} \varphi(b(ab)^n)$.*

At first glance, this definition seems non-intuitive. However, the idea is that we defined $\tilde{G}_a$ so that the functional relation $\tilde{G}_a(b) = G_a(b^{-1})$ carries over. And indeed it does!

Lemma 2.18. *G_a and $\tilde{G}_a$ are related by the functional equation $G_a(b^{-1}) = \tilde{G}_a(b)$.*

Proof. We once again sweep all the convergence results under the rug, so

$$\begin{aligned}\tilde{G}_a(b) &:= \sum_{n=0}^{\infty} \varphi(b(ab)^n) \sum_{n=0}^{\infty} b\varphi((ab)^n) = b \cdot \sum_{n=0}^{\infty} \varphi((ab)^n) = b \cdot \varphi(\textstyle\sum_{n=0}^{\infty}(ab)^n) = b \cdot \varphi((1_A - ab)^{-1}) \\ &= \varphi(b(1_A - ab)^{-1}) = \varphi((b^{-1})^{-1}(1_A - ab)^{-1}) = \varphi((b^{-1} - a)^{-1}) =: G(b^{-1}).\end{aligned}$$

The last equality uses the fact that $(a_1 a_2)^{-1} = a_2^{-1} a_1^{-1}$ for any invertible $a_1, a_2 \in A$. We also quietly assumed that $b \in B$ was itself invertible with a well-defined inverse b^{-1}. ■

Remark 2.19. *Lehner flips the definitions around, interestingly. He refers to $\tilde{G}_a$ as the Cauchy transform, and G_a as the "G-transform". I found this to be less intuitive so I avoided it.*

The main stars of the transforms are, arguably, the R- and K-transforms. Recall that we previously used $R_a(z) = \sum_{n=0}^{\infty} \kappa_{n+1}(a) z^n$, so we co-opt this by replacing z with b, as follows.

Definition 2.20 (R-transform)**.** *Given $a \in A$, define $R_a(b) := \sum_{n=0}^{\infty} \kappa^{(n+1)}(a \otimes_B \underbrace{ba \otimes_B \ldots \otimes_B ba}_{n \text{ times}})$.*

Consequently, define $K_a(b) := b^{-1} + R_a(b)$, where we again assume b^{-1} exists.

Just as before, the R-transform linearises free additive convolution.

Lemma 2.21. *Given self-adjoint $a, a' \in (A, \varphi)$ for a B-valued $*$-probability space, we have that $R_{a+a'}(b) = R_a(b) + R_{a'}(b)$, for all $b \in B$.*

The other important takeaway from these definitions, though, is that the functional relation $G[K(z)] = z = K[G(z)]$ carries over, in some capacity.

Remark 2.22. *Assuming "nice enough conditions", $G_a[K_a(b)] = b = K_a[G_a(b)]$.*

This is currently an incredibly vague statement, but as Lehner and Speicher both point out, the aforementioned *nice enough conditions* still allow a great deal of flexibility. We revisit this remark as a proper theorem in the next section.

With that, the only other preliminaries of FPT yet to be considered is the concept of distributions.

Remark 2.23. *The next subsection, on distributions, is (in hindsight) not relevant for our current purposes. However, they are necessary in order to have a complete treatment of B-valued FPT. Moreover, it might still be worthwhile reading about them, as they seem to be important for generalising the discussion concerning spectra of polynomials of free variables to the operator-valued setting.*

2.6 Distributions

Observe that, in the scalar-valued setting, we can consider the $*$-distribution μ for some fixed $a \in (A, \varphi)$ as a map $\mathbb{C}\langle z, z^* \rangle \to \mathbb{C}$, which determines all the moments of μ via the action of φ on the corresponding polynomial in a ([NS06] p.26).

Example 2.24. *The distribution for a standard semi-circular element s is given by $\mu : \mathbb{C}\langle z, z^*\rangle \to \mathbb{C}$ where, for example, $\mu(z^*) = \mu(z) = 0$, $\mu(z^2) = \mu(z^*z) = \mu((z^*)^2) =$*

We aim to do the same for operator-valued FPT, and define a $*$-distribution as a map resembling "$B\langle X\rangle \to B$". First, we fix some self-adjoint (or, at least, normal) $a \in (A, \varphi)$.

Definition 2.25 (Noncommutative polynomial algebra)**.** *Given a unital $*$-algebra B, define $B\langle X, X^*\rangle := \mathbb{C}\langle X\rangle * B$ to be the space of all B-linear non-commuting polynomials in a variable X.*

One way to visualise $B\langle X\rangle$ is by thinking of elements $b * p(X, X^*)$ as *words* w in our space. Then, $B\langle X\rangle$ consists of (the closure of) all strings of the form $w_1 \cdot w_2$ and $w_1 + w_2$, where $(b_1 * p_1) \cdot (b_2 * p_2)$ is a formal product.

With that in hand, the only other preliminary we must define is the notion of a *character* with respect to a fixed $a \in A$. Informally speaking, this refers to a mapping which replaces all instances of X in $w \in B\langle X\rangle$ with a, and replacing X^* with $a^* = a$. The below definition formalises this intuition.

Definition 2.26 (Characters)**.** *The character $\tau_a : B\langle X\rangle \to A$ is the unique map such that:*

1. *$\tau(b) := b$ for all $b \in B$;*
2. *$\tau(X) := a$ and $\tau(X^*) := a^* = a$;*
3. *$\tau(w_1 \cdot w_2) = \tau(w_1)\tau(w_2)$ for all $w_1, w_2 \in B\langle X\rangle$.*

Example 2.27. *Given a character τ_a, we have $\tau_a(b * X^2) = ba^2$, $\tau_a(b * (X^*X)) = ba^*a$, and so on.*

Example 2.28 (The $(M_d(\mathbb{C})\otimes\mathcal{A}, \mathrm{Id}_d\otimes\tau_{\mathcal{H}})$ setting)**.** *Let $\mathcal{A}_d := M_d(\mathbb{C})\otimes\mathcal{A}$, $\varphi := \mathrm{Id}_d\otimes\tau_{\mathcal{H}}$, and $B := M_d(\mathbb{C})$. Then, for some self-adjoint $a \in \mathcal{A}_d$ (e.g. $a = X_{\text{free}}$), we have:*

$$\tau_a(X^* \cdot (b * X^2) \cdot (b')) = (a^*) \cdot (ba^2) \cdot (b'1_A) = a^*ba^2b'.$$

At this point, we have two maps $\tau_a : B\langle X, X^*\rangle \to A$ and $\varphi : A \to B$. The next step should seem natural!

Definition 2.29 (Operator-valued $*$-distribution)**.** *The distribution of $a \in A$ is defined as $\nu_a : B\langle X\rangle \to B$ given by $\nu_a := \varphi \circ \tau_a$.*

It shouldn't be a surprise what we define as the free additive convolution of $*$-distributions now!

Definition 2.30. *Let $\boxplus$ denote the operation such that $\nu_a \boxplus \nu_{a'} := \nu_{a+a'}$.*

By pulling back the R-transform as a transform with respect to distributions, we get:

Corollary 2.30.1. *The R-transform acts as $R_{\nu\boxplus\nu'}(b) = R_\nu(b) + R_{\nu'}(b)$.*

We extend this definition of a B-valued $*$-distribution for a single element a to that of a joint distribution corresponding to $(a_1, \ldots, a_m)$. This would entail similar definitions for $B\langle X_1, \ldots, X_m, X_1^*, \ldots, X_m^*\rangle$, then defining the character $\tau_{(a_1,\ldots,a_m)}$, and finally $\nu_{(a_1,\ldots,a_m)} := \varphi \circ \tau_{(a_1,\ldots,a_m)}$.

3 Onwards to analysing X_{free}

As a reminder, we are interested in computing $\|X_{\text{free}}\|$ in the C^*-norm for the space $\mathcal{A}_d := M_d(\mathbb{C}) \otimes \mathcal{A}$, where $(\mathcal{A}, \tau_{\mathcal{H}})$ is some full Fock space, and $(\mathcal{A}_d, \varphi)$ is defined as in Example 2.28. By definition, note that X_{free} is non-zero and self-adjoint; hence, $\mathrm{sp}(X_{\text{free}}) \subseteq \mathbb{R}$ and $\|X_{\text{free}}\| = \max|\mathrm{sp}(X_{\text{free}})|$.

To that end, using the notation in Equation (1), Lehner noted the following results [Leh99].

Theorem 3.1. *Given some non-zero self-adjoint $a \in \mathcal{A}_d$, $\tilde{G}_a : \mathbb{H}^+(B) \to G_a(\mathbb{H}^+(B))$ is invertible along the interval $(0, \frac{1}{\|a\|})1_B := \{s1_B : 0 < s < \frac{1}{\|a\|}\}$*

Since $\tilde{G}_a(b)$ is invertible along $(0, \frac{1}{\|a\|})1_B$, it follows that $G_a(b) = \tilde{G}_a(b^{-1})$ is invertible along the interval $(-\infty, \min \mathrm{sp}(a))1_B \cup (\max \mathrm{sp}(a), \infty)1_B$. Since the K-transform is an inverse for G_a, Lehner's main idea is to express the endpoints of the intervals, $\min \mathrm{sp}(a)$ and $\max \mathrm{sp}(a)$ in terms of the range of K_a.

For instance, since $G_a[K_a(b)] = b$, we consider $b \in B$ such that $K_a(b) \in (\max \mathrm{sp}(a), \infty)1_B$. With some rewriting, we get:

$$\begin{aligned} \max \mathrm{sp}(a) &= \inf\{s \in \mathbb{R} \mid s \in (\max \mathrm{sp}(a), \infty)\} \\ &= \inf\{s \in \mathbb{R} \mid \exists b \in B_+ : K(b) = s1_B\} \end{aligned}$$

A similar infimum argument holds for $\min \mathrm{sp}(a)$. The rest of [Leh99] is about explicitly computing this infimum by deriving the K-transform corresponding to X_{free}.

4 Miscellaneous notes (not important)

4.1 The tensor product with respect to B

Technically, it's not $A^{\otimes n}$ that we are considering here, after all! Indeed, the usual tensor product $A \otimes A$ offers no indication that we wish for elements of B to be able to "be pulled out" to the left or to the right, like when we defined the conditional expectation!

In other words, we seek to identify $a \otimes ba'$ and $ab \otimes a'$ to denote the same element in this new tensor product structure.

Definition 4.1 ($\otimes_B$ tensor product)**.** *We define the tensor product with respect to B as $A \otimes_B A := (A \otimes A)/\sim$, where $\sim$ is an equivalence relation such that, for all $a, a' \in A$ and all $b \in B$, we have*

$$a \otimes ba' \sim ab \otimes a'$$

We can then inductively define $A^{\otimes_B n}$ for all $n \in \mathbb{N}$, where $A^{\otimes_B 0} := B$ and $A^{\otimes_B 1} := A$.

4.2 The minimal norm

There has been a minor inconsistency hidden in these notes, when we supposed that $X_{\text{free}} \in M_d(\mathbb{C}) \otimes \mathcal{A}$. In a more thorough treatment (as in [Pis03] pp.1-2), we would have defined $M_d(\mathbb{C}) \otimes_{\min} \mathcal{A}$ as the algebraic closure of $M_d(\mathbb{C}) \otimes \mathcal{A}$ within the space $\mathcal{B}(\mathbb{C}^d \otimes \mathcal{T}(\mathcal{H}))$.

Upon equipping $M_d(\mathbb{C}) \otimes_{\min} \mathcal{A}$ with the induced C^*-norm $\|\cdot\|_{\min}$, we can take $X_{\text{free}} \in M_d(\mathbb{C}) \otimes_{\min} \mathcal{A}$, and consider $\|X_{\text{free}}\|_{\min}$, which is indeed the notation used by Lehner in [Leh99].

4.2.1 The free additive convolution

The following is an alternative formulation of the transforms and additive convolution, as described in Section 10.4.2 of [MS17]). Perhaps this could be better suited for understanding? In this formulation, we forego defining the moment transform entirely.

[TODO: Translate the theorem into our notation.]

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