Overview:

→ Main goal: Understand BBvH 1 and 2.
→ in BBvH, Lemma 2.4 uses Lehner '99, and resolvents (more about them in W5)
→ Lehner '99 talks about free probability theory.
So, here we are trying to first understand FPT.

Review. Last week, Robert described:

1. free cumulants κ_n^a
2. additivity of free cumulants
   - if a, b free, $\kappa_n^{a+b} = \kappa_n^a + \kappa_n^b$.
3. moment-cumulant formula (mcf)
   $$\varphi(a^n) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \kappa_{|V|} \right)$$
   if clear from context, $k_n := \kappa_n^a$.
4. factoring the NC lattice.
   - if $\pi = \{V_1, \dots, V_k\} \in NC(n)$,
     then $[0_n, \pi] = \{\sigma : 0_n \le \sigma \le \pi\}$
     $\hat{=}\ NC(|V_1|) \times \dots \times NC(|V_k|)$
5. Kreweras complement.
   - if $|\pi| = k$, then $|Kr(\pi)| = n-k$, where $\pi \in NC(n)$.
   - $Kr^{2n} = Id$.
   - $Kr \circ Kr \equiv$ ~~left~~ right shift permutation *: not mentioned previously

This time:

0. Cauchy-transform (Nica-Speicher §2.4 pg 40)
I. R-transform. (NS. chp. 12 + chp. 16)
   WARNING: Two different notions of R-transform.
   I will use a def^n consistent with chp 16.
I'. sum of free r.v.
II. multiplicative free convolution.
   (III. applications!)

0'. Example for mcf. $\varphi(a^n) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} k_{|V|} \right)$

- $\varphi(a) = k_1$
- $\varphi(a^2) = k_1^2 + k_2 \Longrightarrow k_2 = \varphi(a^2) - \varphi(a)$ "variance"
- $\varphi(a^3) = k_1^3 + 3k_1k_2 + k_3$

Additional note: When given $\varphi(a^n)$ $\forall n$, mcf uniquely determines k_n.

0. Cauchy transform.

Setup: for the rest of this session, let $a \in A_{sa}$ self-adj. with μ its distribution (compactly supp over $\mathbb{R}$)

Def^n. (Cauchy transform.)
$$G_a \equiv G_\mu : \mathbb{C}_+ \longrightarrow \mathbb{C}_-$$
$$z \longmapsto \int_{\mathbb{R}} \frac{1}{z-t}\, d\mu(t)$$

Fact: G_μ is analytic, with power series expansion on $\{|z| > r\}$ where $r := \sup\{|t| : t \in \operatorname{supp} \mu\}$

Prop^n: $G_\mu(z) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{\varphi(a^n)}{z^n} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{m_n}{z^n}$

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Pf. Recall:
$$\frac{1}{z-t} = \sum_{n=0}^{\infty} \frac{t^n}{z^{n+1}} \quad \text{for all } t \in \operatorname{supp}\mu.$$
and this series converges uniformly. $\square$

$$G_\mu(z) = \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{t^n \, d\mu}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \underbrace{\int_{\mathbb{R}} t^n \, d\mu}_{m_n = \varphi(a^n)}$$

Fact. Given G_μ, can recover μ too!

Def^n. $\forall \varepsilon > 0$, $h_\varepsilon(t) := -\frac{1}{\pi} \operatorname{Im}[G_\mu(t+i\varepsilon)]$

Thm. (Stieltjes inversion).
$$\left.\frac{d\mu}{dt}\right|_{t=t} = \lim_{\varepsilon\to 0^+} h_\varepsilon(t).$$

That's all for now!

$$\mu \rightleftarrows G_\mu$$

I. The R-transform.

Setup: Given $a, b \in A_{sa}$ (self-adjoint) with distributions μ, ν respectively. ($a \rightsquigarrow \mu$, $b \rightsquigarrow \nu$)
We know $a+b \in A_{sa}$, so it also has a distribution that is compactly supported over $\mathbb{R}$.

Want: express this w.r.t. μ, ν.
denote it by $\mu \boxplus \nu$ "free additive convolution" "boxed plus".

Def^n. Transforms.

(Notation: $\mathbb{C}_0[[z]] :=$ set of formal p.s. with zero constant term.)

1. Moment transform:
$$M_a(z) = \sum_{n=1}^{\infty} \varphi(a^n) z^n$$

2. R-transform:
$$R_a(z) = \sum_{n=1}^{\infty} \kappa_n z^n \leftarrow z\mathcal{R}(z) = R(z)$$
$\mathcal{R}$: ch. 12. R: ch. 16.

3. "Cumulant transform"
$$\longrightarrow C_a(z) = 1 + R_a(z) = \sum_{n=0}^{\infty} \kappa_n z^n \quad \text{where } \kappa_0 = 1.$$

Aside: Voiculescu's first motivation for FPT.

$\mu \boxplus \mu$

$\mu^{\boxplus n} \leftarrow \mathbb{R}_{>0}$

$\|$

$(\mu_n)_{n \in \mathbb{R}_{>0}}$

Prop^n: (Cauchy and M_μ.)
$$G_\mu(z) = \frac{1}{z}\left(1 + M_\mu(1/z)\right). \qquad (1)$$

Proof.
$$G_\mu(z) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{\varphi(a^n)}{z^n} \qquad \varphi(1_A) = 1, \quad \varphi(a^0) = 1$$
$$= \frac{1}{z}\left(1 + \underbrace{\sum_{n=1}^{\infty} \varphi(a^n)\cdot\left(\frac{1}{z}\right)^n}\right)$$
$$= \frac{1}{z}\left(1 + M_\mu(1/z)\right) \quad \square$$

Thm: If a, b free, then $R_{a+b} = R_{\mu\boxplus\nu} = R_\mu + R_\nu = R_a + R_b$.

Proof.
Recall $\kappa_n^{a+b} = \kappa_n^a + \kappa_n^b$.
Additivity follows by definition.
$$R_{a+b}(z) = \sum_{n=1}^{\infty} \kappa_n^{a+b} z^n = \sum_{n=1}^{\infty} (\kappa_n^a + \kappa_n^b) z^n = R_a(z) + R_b(z).$$

[illegible] $\xrightarrow{?} \mu \boxplus \nu$

"$R_{a+b}(z) = \sum_{n=1}^{\infty} \kappa_n^{a+b} z^n = \sum_{n=1}^{\infty} (\kappa_n^a + \kappa_n^b) z^n = R_a(z) + R_b(z)$.

Problem:
$$\begin{tikzcd} R_{\mu \boxplus \nu} \arrow[rr, "?"] \arrow[dr, bend right, "?"'] & & \mu \boxplus \nu \\ & G_{\mu \boxplus \nu} \arrow[ur] & \end{tikzcd}$$

Thm: (Functional Equation for R-transform.)

$$R_a\left[z(1+M_a(z))\right] = M_a(z) \;\longleftarrow (\text{FER.})$$

Proof: SOON!!

$$R_a : (M_a(z)+1)z \longmapsto M_a(z)$$

$$\begin{tikzcd} R_a \arrow[r, leftrightarrow, "\text{FER.}"] & M_a \arrow[r, leftrightarrow] & G_a \arrow[r] \arrow[ll, bend left] & \mu \end{tikzcd}$$

Corollary: (Cauchy and R-transform).

1. $G_\mu\left[\frac{1}{z}(\underbrace{1+R_\mu(z)}_{C})\right] = z$.
2. $R_\mu[G_\mu(z)] + 1 = z\,G_\mu(z)$.

proof: (denote by G, R, etc.)

Recall: $C(z) = 1 + R(z)$.

define $K(z) = \frac{1}{z} C(z)$.

WTS 1'. $G[K(z)] = z$. i.e. "K inverts G" as formal p.s.

2'. $K[G(z)] = z$.

$$\begin{aligned} K[G(z)] &= \frac{1}{G(z)} C[G(z)] \quad \text{by (1)} \\ &= \frac{1}{G(z)} C\left[\frac{1}{z}(1+M(1/z))\right] \quad \text{by FER.} \\ &= \frac{1}{G(z)}(1+M(1/z)) \quad \text{by (1)} \\ &= z. \end{aligned}$$

$\longrightarrow$ proves 2'

$\Longrightarrow$ $R\left[\frac{1}{z}(1+M(1/z))\right] = M(1/z)$.

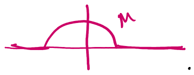
(proof for 2. omitted.)

$z = \cancel{G(z)} = \frac{1}{\cancel{z}} \cdot \frac{(1+M(1/z))}{G(z)}$

Examples

1. Let $a \in A_{sa}$ semi-circular of radius 2. (cf. Week 1)

Then $\mu(t) = \begin{cases} \frac{1}{2\pi}\sqrt{4-t^2}, & |t| \le 2 \\ 0, & \text{else}. \end{cases}$



and $\varphi(a^n) = \begin{cases} 0, & n \text{ odd} \\ C_k, & n = 2k. \end{cases}$

WTS: $R_a(z) = z^2$.

Pf: using uniqueness of k_n given by mcf.

("the proof is by cheating" — Nica)

$k_n = (0, 1, 0, 0, \ldots)$

$k_2 = \varphi(a^2) - \varphi(a)^2 = 1 - 0 = 1$

WTS: $k_m = 0 \quad \forall m > 2$.

$\varphi(a^m) = \sum_{\pi \in NC(m)} \prod_{V \in \pi} k_{|V|} = k_m + \sum_{\pi \in NC_2(m)} 1$

I.H. $k_m = \begin{cases} 0 - 0, & m \text{ odd} \\ C_m - C_m, & m \text{ even} \end{cases} = 0$.

2. Let $a \in A_{sa}$. Then a is "Bernoulli" when $\varphi(a^n) = \begin{cases} 0, & n \text{ odd} \\ 1, & n \text{ even}. \end{cases}$

Fact: $\mu = \frac{1}{2}(\delta_{-1} + \delta_{+1})$.

2. Let $a \in A_{sa}$. Then a is "Bernoulli" when $\varphi(a^n) = \begin{cases} 0, & n \text{ odd} \\ 1, & n \text{ even}. \end{cases}$

Fact: $\mu = \frac{1}{2}(\delta_{-1} + \delta_{+1})$.

Then, $M_\mu(z) = z^2 + z^4 + \dots + z^{2n} + \dots$
$= \sum_{m=1}^{\infty} z^{2m} = \frac{z^2}{1-z^2}$.

$R[z(1+M_\mu(z))] = M_\mu(z)$

Now, $z(1+M_\mu(z)) = \frac{z}{1-z^2}$;

by FER., $R\left(\frac{z}{1-z^2}\right) = \underbrace{\frac{z^2}{1-z^2}}_{M_\mu(z)}$

Change variables $w \leftarrow \frac{z}{1-z^2} \Rightarrow z = \frac{-1+\sqrt{1+4w^2}}{2w}$.

$R(w) = zw$ so $R(w) = \frac{-1+\sqrt{1+4w^2}}{2}$.

So, $R(w) = zw$
$= \frac{-1+\sqrt{1+4w^2}}{2}$. $\Rightarrow \boxed{\begin{aligned} R_{\mu\boxplus\mu}(z) &= 2R_\mu(z) \\ &= -1+\sqrt{1+4z^2} \end{aligned}}$

$\cdot R[G]+1 = zG$

$\Rightarrow R_{\mu\boxplus\mu}[G_{\mu\boxplus\mu}(z)] + 1 = (\cancel{-1} + \sqrt{1+4G_{\mu\boxplus\mu}(z)^2}) \cancel{+1}$
$= z\,G_{\mu\boxplus\mu}(z)$ (by FER.)

$\Rightarrow 1+4G^2 = z^2G^2$

$\Rightarrow G_{\mu\boxplus\mu}(z) = \frac{1}{\sqrt{z^2-4}}$

ready for Stieltjes inversion!

$$\left.\frac{d\mu\boxplus\mu}{dt}\right|_t = -\frac{1}{\pi} \lim_{\varepsilon\to 0} \left(\operatorname{Im} \frac{1}{\sqrt{(t+i\varepsilon)^2-4}} \right)$$
$$= -\frac{1}{\pi} \operatorname{Im} \frac{1}{\sqrt{t^2-4}}$$
$$= \begin{cases} 0, & |t| \geq 2 \\ \frac{1}{\pi\sqrt{4-t^2}}, & |t| < 2. \end{cases}$$

i.e. $\mu \boxplus \mu$ is ...

the arcsine distribution.

(Note: in classical case
$\mu * \mu$ gives Binomial distⁿ)

So arcsine distⁿ
= "free Binomial distⁿ"

3. Let μ be Bernoulli as before.

Want: $\mu^{\boxplus n} = \underbrace{\mu \boxplus \cdots \boxplus \mu}_{n \text{ times}}$.

$$R_{\mu^{\boxplus n}}(z) = n R_\mu(z) = \frac{n}{2}\left(-1+\sqrt{1+4z^2}\right)$$

$\longrightarrow R[G]+1 = zG \Longrightarrow G_{\mu^{\boxplus n}}(z) = \frac{-(n-2)z + \sqrt{z^2 \dots}}{2(z^2-n^2)}$

$$\longrightarrow R[G]+1 = zG \implies G_{\mu \boxplus n}(z) = \frac{-(n-2)z + \sqrt{z^2 \cdots}}{2(z^2-n^2)} \to P(z)/Q(z).$$

General process: (given μ, ν) (or a, b)

1. Find G_μ, G_ν (or M_μ, M_ν)

2. Use (FER) to calculate R_μ, R_ν

3. $R_{\mu \boxplus \nu} = R_\mu + R_\nu$

analytically, if possible

4. Calculate $G_{\mu \boxplus \nu}$

5. Use Stieltjes inversion to recover $\mu \boxplus \nu$.

Proof of (FER.) Recall $R_a(z) = \sum_{n=1}^{\infty} k_n z^n$.

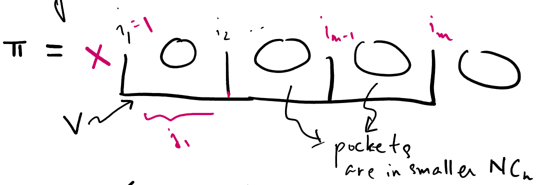
and $\varphi(a^n) = \sum_{\pi \in NC_n} \prod_{V \in \pi} k_{|V|}$

WTS: $[z^n] R_a\left[z(1+M_a(z))\right] = [z^n] M_a(z)$

Then, parametrise $\pi \in NC(n)$ by its left-most block.

i.e. by $V_\pi = \{ i_1, \dots, i_m : 1 = i_1 < i_2 < \dots < i_m \leq n \}$

Picture:



So, $\pi \longleftrightarrow (\underbrace{m}_{\text{size of leftmost block}} ; \underbrace{j_1, \dots, j_m}_{\text{size of pockets}} ; \underbrace{\pi_1, \dots, \pi_m}_{\pi_r \in NC(j_r)})$

size of leftmost block $m = |V|$, $1 \leq m \leq n$

size of pockets $j_r = i_{r+1} - i_r \geq 0$, $j_1 + \dots + j_m = n-m$

$\pi_r \in NC(j_r)$

$\boxed{|NC(0)| = 1}$

Then, substitute into $\varphi(a^n)$ formula.

$$\varphi(a^n) = \sum_{m=1}^{n} \left[\ ? \ \right]$$

$$= \sum_{m=1}^{n} k_m \sum_{\substack{j_1, \dots, j_m \geq 0 \\ j_1 + \dots + j_m = n-m}} \left[\ ? \ \right]$$

$$= \sum_{m=1}^{n} k_m \cdot \sum_{\substack{j_1, \dots, j_m \geq 0 \\ j_1 + \dots + j_m = n-m}} \sum_{\substack{\pi_1 \in NC(j_1) \\ \vdots \\ \pi_m \in NC(j_m)}} \left[\prod_{V \in \pi_1} k_{|V|} \right] \cdots \left[\prod_{V \in \pi_m} k_{|V|} \right]$$

$$= \sum_{m=1}^{n} k_m \cdot \sum_{\vec{j}} \underbrace{\Big(\sum_{\pi\in NC(j_1)} \prod_{V\in\pi} k_{|V|}\Big)}_{\varphi(a^{j_1})} \cdots \underbrace{\Big(\sum_{\pi\in NC(j_m)} \prod_{V} k_{|V|}\Big)}_{\varphi(a^{j_m})}$$

$$\varphi(a^n) = \sum_{m=1}^{n} \sum_{\substack{j_1,\dots,j_m\ge 0 \\ \sum j_r = n-m}} k_m \cdot \prod_{r=1}^{m} \varphi(a^{j_r}) \, .$$

Then, multiply both sides by z^n, and sum over $n\in\mathbb{N}_+$

LHS becomes $M_a(z)$.

And $$[z^n]\, M_a(z) \overset{RHS}{=\!=} z^n \underbrace{\sum_{m=1}^{n} \sum_{\substack{j_1,\dots,j_m\ge 0 \\ j_1+\dots+j_m = n-m}} k_m \varphi(a^{j_1}) \cdots \varphi(a^{j_m})}_{\circledast}$$

Now, $R_a[z(1+M_a(z))]$ expand M_a

$$= R_a\Big[z\Big(1+\sum_{k\ge1}\varphi(a^k) z^k\Big)\Big]$$ expand R_a

$$= \sum_{n'=1}^{\infty} k_{n'}\, z^{n'} \Big(1+\sum_{k\ge1}\varphi(a^k)z^k\Big)^{n'}$$

$$\overbrace{\Big(1+\sum_{k\ge1}\varphi(a^k)z^k\Big)\cdots\Big(1+\sum_{k\ge1}\varphi(a^k)z^k\Big)}^{m}.$$

z^{j_1} $z^{j_2}\cdots$ z^{j_m}

$$= \sum_{n=1}^{\infty} z^n \cdot [\ ?\ ]$$

(idea: $k_m z^m \cdot$ something z^{n-m} !)

$$= \sum_{n=1}^{\infty} \Big[\sum_{m=1}^{n} k_m z^m \cdot \Big(\sum_{\substack{j_1,\dots,j_m\ge 0 \\ j_1+\dots+j_m = n-m}} \prod_{r=1}^{m} \varphi(a^{j_r})\, z^{j_r}\Big)\Big]$$

$$= \sum_{n=1}^{\infty} \circledast \ !$$

proof done.

III. Products of r.v.s & free mult. convolution.

Recall the Kreweras complement. (denoted Kr or K).

Def. (Free mult. convolution) Given $f,g\in\mathbb{C}_0[[z]]$

s.t. $f(z) = \sum_{n=1}^{\infty}\alpha_n z^n$, $g(z) = \sum_{n=1}^{\infty}\beta_n z^n$,

$$f \boxtimes g\,(z) := \sum_{n=1}^{\infty} \gamma_n z^n ,$$

where $$\gamma_n := \sum_{\pi\in NC(n)} \Big(\prod_{V\in\pi}\alpha_{|V|}\Big)\Big(\prod_{W\in Kr(\pi)}\beta_{|W|}\Big)$$

$z^{|V_1|}\cdots z^{|V_k|}$ $z^{|W_1|}\cdots z^{|W_{n-k}|}$

(1960s): → posets.

Examples: (small n).

- $\gamma_1 = \sum_{\pi\in NC(1)}(\cdots) = \alpha_1\beta_1$
- $\gamma_2 = \sum_{\pi}(\cdots) = \alpha_2\beta_1 + \alpha_1\beta_2$

K ⊔ K ||

Properties: 1. $\boxtimes$ is associative $\rightsquigarrow$ fact.

2. $\boxtimes$ is commutative

3. Let $Id(z) = z$. Then Id is mult. unit wrt. $\boxtimes$

~~[Corollary: $(\mathbb{C}_0[[z]], \boxtimes)$ is monoid.]~~

$f \boxtimes Id = f = Id \boxtimes f$.

proof: "by computation."

$\rightarrow$ (not hard to show 2., 3.)

Def: Let $\zeta(z) := \sum_{n=1}^{\infty} z^n$ (actually ζ, but I can't write well.)

ζ related to ζ from Wk 3.

Observe "Fact": 1. $M_a = R_a \boxtimes \zeta \longrightarrow$ (!!)

$\underset{m_n}{\alpha_n} = k_n * \zeta$ ← posets $NC(n)$

2. ζ is invertible wrt. $\boxtimes$, $\quad \zeta \boxtimes Möb = Id.$

$$\underbrace{\zeta^{-1} =: Möb := \sum_{n=1}^{\infty} \underbrace{(-1)^{n-1} C_n}_{\text{Möbius}} z^n}_{\text{Möbius inversion!}}$$

$f \boxtimes g \rightarrow$ a,b free ab.

Remarks. 1. if $a, b \in A_{sa}$, then $ab \notin A_{sa}$ (not necessarily).

$$\underset{x}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \underset{z}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \notin A_{sa}.$$

So, we restrict to $a, b \in A_+ \Longrightarrow ab \in A_+ \quad sp(a) \subseteq \mathbb{R}_{\geq 0}$.

Rmk: (actually, just one suffices ($\mu \subseteq \mathbb{R}_+$), but symmetry makes it simpler).

2. We can generalise to all of $a, b \in A_{sa}$, but: i. it's not worth it

ii. it's not of use to us.

~~3. Fact: $Möb(z(1+z)) = z$~~

(End of remarks)

Recall: factorisation of NC Lattice.

If $\pi = \{V_1, \dots, V_k\} \in NC(n)$, then $[0_n, \pi] = \{\sigma \in NC(n) : 0_n \leq \sigma \leq \pi\}$
$\cong NC(|V_1|) \times \dots \times NC(|V_k|)$

Thm. (Multiplicative convolutions of R-transform).

a,b free. 1. $M_{ab} = R_a \boxtimes M_b \longrightarrow (M_{ab} = M_a \boxtimes R_b)$

2. $R_{ab} = R_a \boxtimes R_b$.

1. $M_{ab} = R_a \boxtimes M_b$

2. $R_{ab} = R_a \boxtimes R_b$.

(Note: If 1., then $\underbrace{M_{ab} \boxtimes M\ddot{o}b}_{R_{ab}} = R_a \boxtimes \underbrace{M_b \boxtimes M\ddot{o}b}_{R_b} \Rightarrow 2.$)

(Note: If 2., then $\underbrace{\zeta \boxtimes R_{ab}}_{M_{ab}} = \underbrace{\zeta \boxtimes R_a}_{M_a} \boxtimes R_b \Rightarrow 1.$)

So, 1. $\Leftrightarrow$ 2.

Proof. We prove 1. $\Rightarrow$ $M_{ab} = R_a \boxtimes M_b$.

$$M_{ab}(z) = \sum_{n=1}^{\infty} \varphi((ab)^n) z$$

$\varphi(ab\,ab \cdots ab)$

$$\varphi(abab) = \varphi(a)^2\varphi(b^2) + \varphi(a^2)\varphi(b)^2 - \varphi(a)^2\varphi(b)^2.$$

$$\varphi((ab)^n) = \sum_{\sigma \in NC(2n)} \underbrace{\prod_{u \in \sigma} \kappa_u((a,b,a,\dots,b)|_u)}_{\Downarrow}$$

$= 0$ unless $u \subseteq$ odd, $u \subseteq$ even.

$\sigma \in NC(2n)$

π^{odd}, ρ^{even}

$NC_{o/e}(2n) :=$ noncrossing partitions of the form $\rho^{even} \sqcup \pi^{odd}$

Given $\pi \in NC(n)$, $\pi^{odd} := \{2V-1 : V \in \pi\}$

$\pi^{even} := \{2V : V \in \pi\}$.

$$\varphi((ab)^n) = \sum_{\sigma \in NC_{o/e}(2n)} \left(\prod_{V \in \pi^{odd}} \kappa_{|V|}(a,\dots,a)\right)\left(\prod_{W \in \rho^{even}} \kappa_{|W|}(b,\dots,b)\right)$$

$||$

$\sigma = \pi^{odd} \sqcup \rho^{even}$

Claim: $\rho^{even} \leq K(\pi^{odd})$

$$\sum_{\pi \in NC(n)} \sum_{\rho \leq K(\pi)} \left(\prod_{V \in \pi^{odd}} \kappa^a_{|V|}\right)\left(\prod_{W \in \rho^{even}} \kappa^b_{|W|}\right).$$

$$= \underbrace{\sum_{\pi \in NC(n)} \left(\prod_V \kappa^a_{|V|}\right) \cdot \left[\sum_{\rho \leq Kr(\pi)^{w}} \prod \kappa^b_{|W|}\right]}$$

want. $\varphi(b^{|Y_1|}) \cdots \varphi(b^{|Y_q|})$ $\leftarrow$ $M_{ab} = R_a \boxtimes M_b$

fix $\pi \in NC(n)$. Let $K(\pi) = \{Y_1, \dots, Y_q\}$.

$(\rho \leq)\ [0_n, K(\pi)] \cong NC(|Y_1|) \times \cdots \times NC(|Y_q|)$.

$\rho \longmapsto (\rho_1, \dots, \rho_q)$

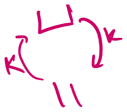
$$\sum_{\rho \leq K(\pi)} \prod_{W \in \rho} \kappa^b_{|W|} = \sum_{\substack{\rho_1 \in NC(|Y_1|) \\ \vdots \\ \rho_q \in NC(|Y_q|)}} \left(\prod_{W_1 \in \rho_1} \kappa^b_{|W_1|}\right) \cdots \left(\prod_{W_q \in \rho_q} \kappa^b_{|W_q|}\right)$$

$$= \underbrace{\left[\sum_{\rho_1} \prod_{W_1} \kappa^b_{|W_1|}\right]}_{\varphi(b^{|Y_1|})} \cdots \left[\sum_{\rho_q} \prod_{W_q} \kappa^b_{|W_q|}\right] \cdots \varphi(b^{|Y_q|}).$$

Examples. 1. $\varphi(ab) = \varphi(a)\varphi(b)$.

2. $\varphi(abab) = \varphi((ab)^2) \rightsquigarrow Y_2$

$$Y_2 := \sum_{\pi \in NC(2)} \left(\prod_{V \in \pi} \kappa^a_{|V|}\right)\left(\prod_{W \in K(\pi)} \varphi(b^{|W|})\right).$$



$\varphi(a\,b\,a\,b) =$

$$\gamma_2 := \sum_{\pi \in NC(2)} \Big(\prod_{V\in\pi} \kappa^a_{|V|}\Big)\Big(\prod_{W\in K(\pi)} \varphi(b^{|W|})\Big)$$

$$= \kappa^a_2\, \varphi(b)^2 + (\kappa^a_1)^2\, \varphi(b^2).$$

$$= [\varphi(a^2) - \varphi(a)^2]\,\varphi(b)^2 + \varphi(a)^2\varphi(b^2)$$

$$= \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2) - \varphi(a)^2\varphi(b)^2.$$

IV. Actual use cases.!!

$\longrightarrow$ Alternate proof of free central limit thm.

Thm. Let $\{a_i\}_{i\in\mathbb{N}}$ be free, identically distrib. s.t. $\varphi(a_i) = 0$, $\varphi(a_i^2) = \sigma^2$

Then,

$$\frac{a_1 + \dots + a_N}{\sqrt{N}} \xrightarrow[N\to\infty]{\text{dist.}} s \leftarrow \text{semicircular}$$

Proof. $\frac{1}{z} R(\text{LHS}) = \frac{1}{z} R(s)$. $\qquad \left[R_{\lambda a}(z) = \lambda R_a(\lambda z)\right]$

$p(x_1, \dots x_n)$

Moral of the tale: Can we now "freely" discuss polynomials of free r.v.s $x_1, \dots, x_n$?

used in
- Chen, Jorge-Vargas, Tropp, van Handel
- Bordenave – Collins
-

Example:

1. Let $\{a_1, a_2\}$, $\{b_1, b_2\}$ free where a_1, a_2, b_1, b_2 are all self-adj.

Consider $p = a_1 b_1 a_1 + a_2 b_2 a_2$ polynomial.

then, p self-adjoint also. $\rightsquigarrow$ role of matrices in FPT.

Q: What is its distribution?

rick: Note that the distribution of

$$\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} =: M(p)$$

is $\mu_{M(p)} = \frac{1}{2}\mu_p + \frac{1}{2}\delta_0$

where the $*$-probability space is

$$\left(M_2(A), \tfrac{1}{2}\mathrm{Tr}\otimes\varphi\right).$$

Now, $M(p) = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}$

So by tracial property:

$\mu_{M(p)} = \mu_{AB}$ where

$$AB := \underbrace{\begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}}_{B}$$

Are we done? Are A, B free?

No! It turns out we need to develop a theory for operator-valued cumulants to address this problem.

It is exactly this theory which leads us directly to Lehner '99.

— thanks for your time!