How to Cut Cake

An Overview of Fair Online Resource Allocation Problems

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Presentation Overview

Breaking it down

Fair Online Resource Allocation Problems

Components

1. "Fair"
2. "Online"
3. "Resource allocation"

Allocation problems

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4. Objective we shall take this to be the net worth of the allocation, subject to *fairness*

Online algorithms

Intuitive idea

A model of algorithms accepting an input instance given as an unknown sequence of inputs (agents, in this case).

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A model of algorithms accepting an input instance given as an unknown sequence of inputs (agents, in this case). After each input agent is presented, the algorithm makes a decision (*irrevocably*, in this case).

(Minimal) online cake-cutting

Defining the problem

Informally...

Congratulations! Today is your birthday so you take a cake into the office to share with your colleagues at tea time. However, as some people have to leave early, you cannot wait for everyone to arrive before you start sharing (allocate) the cake. How do you proceed fairly?

— Toby Walsh (Online Cake Cutting, 2011)

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Simplification

cake $\to I = [0, 1]$;

Allocation

Cutting

If S is a finite set of closed intervals, then:

1. S is a cutting;
2. $\forall [a, b] \in S$ and $c \in (a, b)$, the set $S \cup \{[a, c], [b, c]\} \setminus [a, b]$ is a cutting.

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Simple allocation

An allocation using only n disjoint intervals.

Agent preferences

Valuation

For each $j \in [n]$, define the valuation of agent j denoted by $v_j : 2^I \to \mathbb{R}_{\geq 0}$ given by $v_j(J) = \int_J f_j$ where f_j is a piecewise continuous value density function, such that, for all $j \in [n]$:

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- additive: for any two closed disjoint sub-intervals X, Y, $v_j(X \sqcup Y) = v_j(X) + v_j(Y)$

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Set valuation

For a finite set of intervals S, we define, for all $j \in [n]$, $v_j(S) = \sum_{[a,b] \in S} v_j([a,b])$; in particular, v_j is additive.

Agent preferences

Space of possible valuations

Let $\tilde{C}^0$ denote the space of piecewise continuous functions on I.
Let $\mathscr{V}$ denote the set of possible valuations $v = (v_1, \ldots, v_n)$ where $v_j = \int f_j$, for $f_j \in \tilde{C}^0$.

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Algorithm output

Given some algorithm $\textsc{Alg}$ for this problem and valuations $v_1, \ldots, v_n$, denote the resulting allocation by $A = (A_1, \ldots, A_n) = \textsc{Alg}(v_1, \ldots, v_n)$.

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Complexity

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Robertson-Webb

Two oracles for each $j \in [n]$ as follows

1. $Eval_j(x, y)$ returns the value $v_j([x, y])$, where $[x, y] \subseteq I$;
2. $Cut_j(x, \alpha)$ returns the value $y \in [x, 1]$ such that $v_j([x, y]) = \alpha$.

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$$\forall j \in [n], v_i(A_i) \geq 1/n$$

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$$\forall i \in [n], \forall w_i \in \mathscr{V}_i,$$
$$v_i(f(v_1, \ldots, v_i, \ldots, v_n)) \geq v_i(f(v_1, \ldots, w_i, w_n))$$

Online fairness criteria

Lemma

No envy implies proportionality.

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Lemma

No online cake cutting algorithm is proportional, envy-free, or equitable

Proof.

Suppose some agent i leaves before agent n arrives. A_i is then independent of v_n. If $v_n(A_i) = 1$, agent n will not value any allocation outside A_i. So, not proportional. Since no envy implies proportionality, not envy-free either.

Suppose allocation was equitable, so all agents receive some cake they value. Again, A_i is independent of v_n for the first leaving agent i. □

Online fairness criteria

Online proportionality

Weak proportionality

Each agent j is assigned at least r/k of the total value of the cake to their pieces where

1. r is the value of the remaining amount of unallocated cake when agent j arrives;
2. k is the number of agents yet to be allocated cake at this point.

Online fairness criteria

Online no envy

1. Weakly envy-free: agents do not value cake allocated to agents after their arrival more than their own;
2. Immediately envy-free: agents do not value cake allocated to any agent after their arrival and before their departure more than their own

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Lemma

No envy implies weakly envy-free. Weakly envy-free implies immediately envy-free.

Online fairness criteria

Online equitability

First-come-first-serve

No agent's value of their assigned share can decrease if they arrive earlier in the input sequence and all other agents are left in the same relative positions; formally defined as *arrival monotone* or *meta-envy*).

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More generally, "no agent can profit by a change in their arrival order"

$$\forall \{i, j\} \in \binom{[n]}{2},$$

$$v_i(f_i(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n)) \geq v_i(f_j(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n))$$

Lemma

Equitability implies arrival monotonicity.

Cut-and-choose algorithm

Each application shall [...] allow two mining operations. The Authority shall designate which part is to be reserved solely for the conduct of activities by the Authority.
— UN Convention on the Law of the Sea

Cut-and-choose algorithm

Algorithm 1 I cut but you choose

```
procedure CUT-AND-CHOOSE
    for j = 1 → n−1 rounds do
        The earliest arriving agent so far j cuts the remaining cake once, creating two disjoint intervals X, Y with X ⊔ Y = I_j.
        The second earliest arriving agent so far j+1 chooses whether to take X and leave, or give X to the cutting agent who leaves.
        I_{j+1} ← Y.
    end for
    The last remaining agent takes the leftover cake.
end procedure
```

Implementing cut-and-choose

Using Robertson-Webb oracles

Each iteration of the loop makes one oracle query to the *Cut* oracle, and another to the *Eval* oracle, so overall query complexity is $\Theta(n)$.

Fairness of cut-and-choose

Lemma

The online cut-and-choose procedure is weakly proportional and immediately envy free. However, it is not weakly envy free, equitable, truthful, or arrival monotonic.

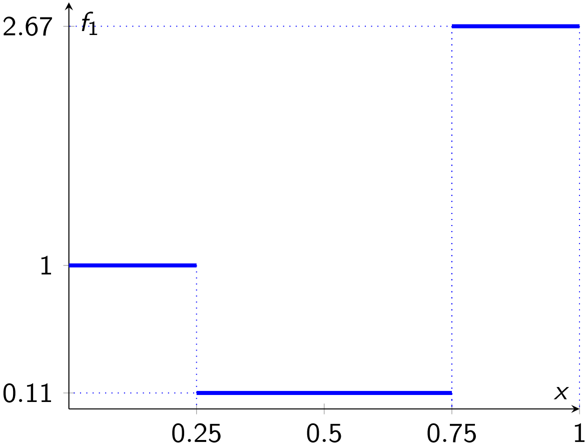
Proof.

Suppose agent i cuts a slice c_i. If allocated the slice, they would want $v_i(c_i) \geq r/k$. But, if not allocated this piece, they would want $v_i(c_i) \leq r/k$. Thus, the best option is to choose $v_i(c_i) = r/k$.

By generalization, this holds for all i, so this is weakly proportional. Also, trivially immediately envy-free.
Consider the following counter-example with 4 agents in the order $1 \to 2 \to 3 \to 4$ for the negative results. $\square$

Fairness of cut-and-choose (contd.)

Proof.



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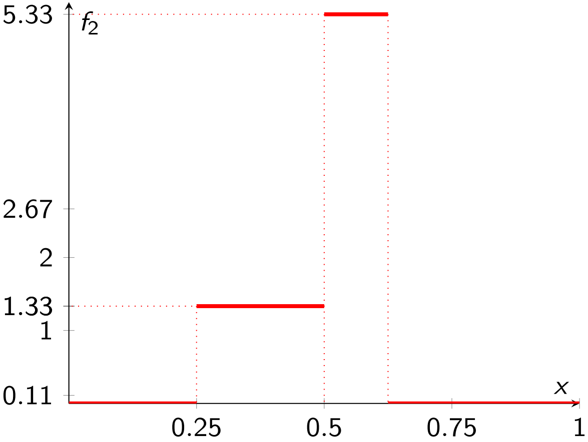

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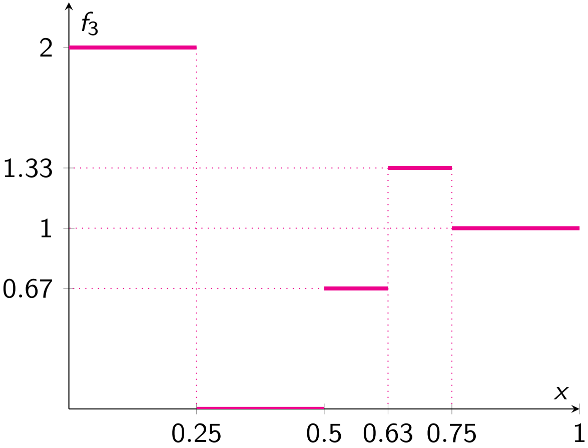

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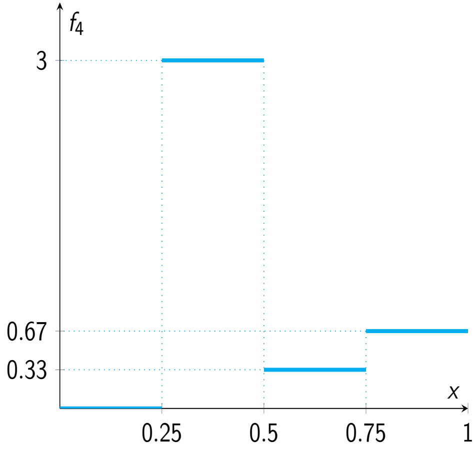

□

Limitations of cut-and-choose

In addition to the negative results mentioned above...

1. We must know n, i.e., the total number of agents partaking in the cake. If n is unknown to the algorithm, then the last agent will be forced into a deadlock.
2. Each agent may have to wait an unspecified amount of time until the next agent shows up, or until they can finally leave with a slice of cake.
3. In particular, since cut-and-choose is not arrival monotone, every agent will prefer to be the second agent instead of the first.

Discounted cut-and-choose

Additional assumptions

1. Each agent $j \in [n]$ has some discount factor δ_j.
2. We also have a maximum patience $T > 0$ which is common for all agents.
3. We have a clock available near the cake, so each agent can note their arrival time.
4. We have a counter near the cake, which each agent, upon their arrival, presses to increment by one.
5. The total number of agents is unknown to all agents (and to the algorithm), but the maximum possible number is n.

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Arrival distribution

In particular, it is reasonable to assume that the arrival of agents is given by some truncated Poisson distribution with some parameter λ. Let λ be given.

Preliminary results

Lemma

For any interval of duration T, the probability of an agent arriving in this interval is $\lambda T + h(T)$, where $h(T) \to 0$ as $T \to 0$.

Let T be such that λT is sufficiently large.

Notation

Define $S_{i,j}$ for $i, j \in [n]$ as

$$S_{i,j} = \frac{(1-\delta_j)\delta_j^{n-1-i}}{2-\delta_j-\delta_j^{n-i}}$$

Modified algorithm

Outline

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1. They note the number of agents already arrived and increment the counter, say it now reads k. First, suppose $k \neq n$.

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3. If there is no agent waiting, agent j cuts a slice C' from the remaining cake C s.t. $v_j(C') = S_{k,j} v_j(C)$. They then wait for time T.

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5. If another agent arrives, then the procedure restarts accordingly.
6. If $k = n$ and no agent waiting, then run off with the cake!
7. If an agent is waiting, then agent j chooses a slice greedily.

Fairness and limitations

Theorem

The above procedure is immediately envy free, arrival monotone, and truthful. It is not weakly envy free or equitable.

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Limitations

1. Nefarious agents may be encouraged to leave before time T has passed.
2. Nefarious agents may delay cutting the cake until very little of the duration T remains.

Online moving knife

1. several rounds of cutting ($n-1$ rounds for minimal cutting)
2. in each round, the algorithm moves a knife from the left to the right, and only stops when some agent declares it to stop
3. at that point, the algorithm cuts the cake and that agent leaves with their share of cake, i.e., the part to the left of the cut.

Offline Dubins-Spanier in Robertson-Webb

Algorithm 2 Offline Dubins-Spanier

procedure DUBINS-SPANIER
    $x_1 \leftarrow 0$; $I_1 \leftarrow [x_1, 1]$; $S_1 \leftarrow [n]$.
    for $j = 1 \rightarrow n-1$ rounds **do**
        $i^* \leftarrow \arg\min\{Cut_i(x_j, 1/n) : i \in S_j\}$
        $x_{j+1} \leftarrow Cut_{i^*}(x_j, 1/n)$
        Allocate $[x_j, x_{j+1}]$ to agent i^*.
        $I_{j+1} \leftarrow [x_{j+1}, 1]$; $S_{j+1} \leftarrow S_j \setminus \{i^*\}$.
    end for
    The last remaining agent takes the leftover cake.
end procedure

Online Dubins-Spanier

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Lemma

The online moving knife procedure is weakly proportional and immediately envy free. However, it is not (weakly) envy free or arrival monotonic.

Dubins-Spanier procedure

Generalized Dubins-Spanier Theorem.

Theorem

Consider a set S and n agents, and let $\mathbb{U}$ be a σ-algebra on S. Suppose each agent j has a countably-additive and nonatomic value measure $v_i : \mathbb{U} \to \mathbb{R}$. Let K be a k-partition of S. Then, the set of all $n \times k$ matrices $[M]_{ij}$ is a compact and convex set in the space of all real-valued $n \times k$ matrices.

Competitive analysis

Utilitarian measure

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For any offline algorithm, the sum cannot exceed n, since $v_j(A_j) \leq 1$ for all $j \in [n]$.
The competitive ratio is then $O(n^2)$. We can construct examples to show tightness.

Truthfulness

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Open question

With what restrictions can we sacrifice randomness without losing minimalism?

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Wikipedia page has a nice summary!

Collusion

Walsh and other papers also look at cases where agents can fix strategies amongst themselves, colluding to get more valuable pieces of cake. In particular, the online-cut-and-choose protocol is resistant to collusion, but the online Dubins-Spanier is not.

Other query models

Simultaneous encoding

All agents succinctly report their discretized value allocations on arrival. This helps with moving from query complexity to space complexity.

More variations of resource allocation

1. Multi-cake
2. Homogeneous goods
3. Indivisible goods
4. Combinatorial auctions

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The End

Questions? Comments?