These notes were compiled for self-reflection only, so that I can look back at this proof in the future. Please excuse the lack of quality as these notes haven't been scrutinised thoroughly, and please suggest making corrections if you see any errors.

These notes were based on some early sections in [Str20].

1 Motivating example

The goal of continuous functional calculus is to express a continuous function on a C^*-algebra in terms of the spectra of the elements in the C^*-algebra. Since this goal might seem rather abstract and impractical, here is an example of an application of such a technique. In this example, we consider the problem of exponentiating a normal matrix $T \in M_n(\mathbb{C})$, i.e., compute $f(T) := \exp(T)$.

Recall that $f(z, \overline{z}) = \exp(z)$ has the uniform limit $p_m \rightrightarrows f$, where, for all $m \in \mathbb{N}$, $p_m(z, \overline{z}) = \sum_{k=0}^{m} \frac{1}{k!} z^k \in \mathbb{C}[z, \overline{z}]$. To carry this forth to a operator on a C^*-algebra $\mathcal{A}$, we seek for $\mathcal{A}$ to be commutative, analogous to the commutativity of $\mathbb{C}[z, \overline{z}]$. Moreover, we wish for the construction to be equivalent to replacing z with T and $\overline{z}$ with T^*. As such, a sufficient condition to assume is that T is normal, and simply consider $C^*(T, \mathrm{Id}_n) \subseteq M_N(\mathbb{C})$, which would then be commutative.

Now, using the results of continuous functional calculus, we wish to write

$$\begin{aligned} f(T) = \exp(T) &= \lim_m p_m(T) \\ &= \lim_m \int_{\mathrm{sp}(T)} p_m(\lambda) d\delta_\lambda && \text{(by spectral mapping theorem)} \\ &= \int_{\mathrm{sp}(T)} \lim_m p_m(\lambda) d\delta_\lambda && \text{(uniform convergence)} \\ &= \sum_{k=1}^{n} \lim_m p_m(\lambda_k) v_k v_k^* && \text{(spectral theorem for normal matrices)} \\ &= \sum_{k=1}^{n} \exp(\lambda_k) v_k v_k^* \end{aligned}$$

2 Overview

Recall that the Stone-Weierstraß theorem allows us to express any continuous function f with compact support as the uniform limit of a sequence of polynomials. Thus, for our purposes, it suffices to construct polynomials over a algebra (a commutative algebra, to be precise, for the same reasons encountered in our example).

Throughout this note, we have the following standing assumptions. Let A be a unital C^*-algebra, and suppose $a \in \mathcal{A}$ is normal ($aa^* = a^*a$). Let $A_0 = C^*(a, 1_{\mathcal{A}})$ denote the unital C^*-subalgebra generated by a and 1_A. Since $A_0 = \mathrm{cl}(\mathrm{span}\{a^n(a^*)^m : m, n \in \mathbb{N}\})$, it follows that A_0 is commutative (where $\mathrm{cl}(X)$ denotes the norm closure of X).

As a remark, we reserve the term *'functional'* for when a map f has (some subset of) $\mathbb{C}$ as its codomain. We denote by $C(X)$ the set of all continuous functionals on a Hausdorff space X, equipped with a norm. Similarly, we denote the set of all continuous functionals with compact support by $C_0(X) \subseteq C(X)$. When X itself is compact, it follows that $C_0(X) = C(X)$. Note that $C_0(X)$ is always a C^*-algebra under the operator norm.

3 The details

3.1 Characters

Definition 3.1 (Character)**.** *A character τ on A_0 is a non-zero unital algebra homomorphism $\tau : A_0 \to \mathbb{C}$. Denote the space of all characters on A_0 by $\Omega(A_0)$.*

In other words, given some $\tau \in \Omega(A_0)$, we must have $\tau(b_1 + b_2) = \tau(b_1) + \tau(b_2)$ and $\tau(b_1 b_2) = \tau(b_1)\tau(b_2)$, and there exists some $b \in A_0$ such that $\tau(b) \neq 0$. Henceforth, we use the shorthand Ω to denote $\Omega(A_0)$.

Note that, once you specify how some $\tau \in \Omega$ acts on a by setting some $\tau(a) = \zeta \in \mathbb{C}$, then you specify the behaviour of τ on all of A_0. To prove this fact, first observe that $\tau((c + id)^*) = \overline{\tau(c + id)}$ for all self-adjoint $c, d \in \mathcal{A}_0$.

We shall also take as a fact that Ω is a non-empty compact Hausdorff space; in particular, $C(\Omega) = C_0(\Omega)$. Some more useful results are as follows.

Lemma 3.1. *Given any $b \in A_0$ and any $\tau \in \Omega$, we have* $\mathrm{sp}(b) = \{\tau(b) : \tau \in \Omega(A)\}$.

Proof. We only prove the $\supseteq$ inclusion here. Suppose, for contradiction, that $\tau(b) \notin^(b)$ for some τ. Then, there exists some $c \in A_0$ such that $(b - \tau(b)1_A)c = 1_A$. Since τ is a character, we get the contradiction

$$1 = \tau(1_A) = \tau((b - \tau(b)1_A)c) = \tau((b - \tau(b)1_A))\tau(c) = (\tau(b) - \tau(b))\tau(c) = 0.$$

The reverse inclusion requires some more abstract algebra, which we needn not worry about. ■

Lemma 3.2. *Given any $\tau \in \Omega$, we have $\|\tau\| = 1$ in the operator norm.*

Proof. Consider some $b \in A_0$ with $\|b\| \leq 1$. Recall that $r(b) = \sup\{|\lambda| : \lambda \in \mathrm{sp}(b)\} \leq \|b\|$, with equality holding precisely when $b = b^*$ is self-adjoint. Since $\tau(b) \in \mathrm{sp}(b)$ from the previous lemma, we get $\tau(b) \leq \|b\| \leq 1$. Thus, $\|\tau\| = \sup\{|\tau(b)| : \|b\| \leq 1\} \leq 1$; since $\tau(1_A) = 1$, we have equality. ■

We shall revisit character spaces once we have defined the Gelfand transform in the following section.

3.2 The (first) Gelfand-Naimark theorem

In this next section, we see a characterisation of commutative C^*-algebras A_0, where we show they are always $*$-isomorphic to $C_0(X)$, for some desired space X.

Definition 3.2 (Linear functionals)**.** *Given a unital C^*-algebra A, let $A^* := \{\phi : A \to \mathbb{C} \mid \phi \text{ is linear}\}$ denote the set of all linear functionals on A.*

We shall claim, without proof, that A^* separates points in A. In other words, for any distinct $b_1, b_2 \in A$, there is some $\phi \in A^*$ such that $\phi(b_1) \neq \phi(b_2)$.

Definition 3.3 (Gelfand transform)**.** *The Gelfand transform is the map $\Gamma : A_0 \to C_0(\Omega)$ given by $b \mapsto \hat{b}|_\Omega$. Here, $\hat{b} : A_0^* \to \mathbb{C}$ is defined by $\hat{b}(\phi) = \phi(b)$ for any linear functional $A_0^* \ni \phi : A_0 \to \mathbb{C}$.*

For convenience, we use the shorthand $\hat{b}$ to denote $\hat{b}|_\Omega$. We call the function $\hat{b}$ the *Gelfand transform of* b. Here are some useful properties of the Gelfand transform.

Lemma 3.3 (Gelfand representation)**.** *The Gelfand transform is a $*$-homomorphism, i.e., the following properties hold.*

1. $\widehat{b_1 + b_2} = \hat{b_1} + \hat{b_2}$;
2. $\widehat{b_1 b_2} = \hat{b_1}\hat{b_2}$;
3. $\hat{b^*} = \hat{b}^*$.

Proof. The proof is by direct computation. Let $\tau \in \Omega$ be arbitrary.

1. $\widehat{b_1 + b_2}(\tau) = \tau(b_1 + b_2) = \tau(b_1) + \tau(b_2) = \hat{b_1}(\tau) + \hat{b_2}(\tau)$;
2. $\widehat{b_1 b_2}(\tau) = \tau(b_1 b_2) = \tau(b_1)\tau(b_2) = \hat{b_1}(\tau)\hat{b_2}(\tau) = (\hat{b_1}\hat{b_2})(\tau)$;
3. $\hat{b^*}(\tau) = \tau(b^*) = \overline{\tau(b)} = \overline{\hat{b}(\tau)} = \hat{b}^*(\tau)$.

■

Theorem 3.1. *For any $b \in A_0$, we have $r(b) = \|b\|$ and $\operatorname{sp}(b) = \{\hat{b}(\tau) : \tau \in \Omega\}$.*

Proof. We have $\|\hat{b}\| = \sup\{|\hat{b}(\tau)| : \|\tau\| = 1\}$. Using Lemma 3.1, Lemma 3.2 and $\hat{b}(\tau) = \tau(b)$, this yields $\|\hat{b}\| = \sup\{|\tau(b)| : \tau \in \Omega\} = \sup\{|\lambda| : \lambda \in \operatorname{sp}(b)\}$. The second statement is a rewording of Lemma 3.1. ■

As a corollary, we also note that the Gelfand transform is norm-decreasing. However, we can get an even better result!

Lemma 3.4. *The Gelfand transform $\Gamma : b \mapsto \hat{b}$ is isometric (i.e., norm-preserving).*

Proof. Note the following computation

$$\|\hat{b}\|^2 = \|\hat{b}^*\hat{b}\| = \|\widehat{b^*b}\| = r(b^*b) = \|b^*b\| = \|b\|^2.$$

■

Finally, we have all the pieces required to assemble the Gelfand-Naimark theorem for commutative C^*-algebras.

Theorem 3.2 (Gelfand-Naimark)**.** *The map* Γ *is an isometric* $*$*-isomorphism.*

3.3 Continuous functionals of characters

With the Gelfand transform as a tool, let's go back to character spaces to construct a homeomorphism $\Omega \to \mathrm{sp}(a)$.

Theorem 3.3. *The Gelfand transform of* a*, i.e.* $\hat{a} : \Omega \to \mathrm{sp}(a)$*, given by* $\hat{a}(\tau) = \tau(a)$ *is a homeomorphism (i.e. a continuous bijection with continuous inverse).*

Proof. That $\hat{a}$ is a bijection follows from the identification in Lemma 3.1; thus we also conclude the existence of the inverse map. Continuity of $\hat{a}$ follows from its linearity. Since Ω is compact, the inverse Gelfand transform maps compact sets onto compact sets, and is hence, itself continuous. ■

We can then lift this homeomorphism $h = \hat{a}$ to a $*$-isomorphism on the algebras of continuous functionals, where we simply *pull back* a function via h. More precisely, we express

Lemma 3.5. *There exists a* $*$*-isomorphism* $\psi : C(\mathrm{sp}(a)) \to C(\Omega)$ *given by* $\psi : f \mapsto f \circ h$.

The proof can be obtained by observing the following diagram of how $\psi(f)$ is computed for all $f \in C(\mathrm{sp}(a))$.

$$\Omega \xrightarrow{h} \mathrm{sp}(a) \xrightarrow{f} \mathbb{C}$$

3.4 Putting it together

Here is a picture of what we have developed so far.

$$C^*(a, 1_A) \xrightarrow{\Gamma} C(\Omega(C^*(a, 1_A))) \xleftarrow{\psi} C(\mathrm{sp}(a))$$

Theorem 3.4 (Continuous functional calculus)**.** *Given a unital* C^**-algebra* A*, and a normal* $a \in A$*, there exists an isometric (norm-preserving)* $*$*-isomorphism*

$$\gamma : C(\mathrm{sp}(a)) \to C^*(a, 1_A); \qquad \gamma : \mathrm{Id} \mapsto a.$$

Proof. Define $\gamma := \Gamma^{-1} \circ \psi$, as the composition of two isometric $*$-isomorphisms. [Alternatively, consider $f : \mathrm{sp}(a) \to \mathbb{C}$ given by $f(z) = z$. Then, $C(\mathrm{sp}(a))$ is generated by f, and γ is the unique unital $*$-homomorphism such that $\gamma(f) = a$, proving injectivity. Since $\gamma(f)$ generates $A_0 = C^*(a, 1_A)$, γ is surjective onto A_0 – an isomorphism!] ■

Often, for any $f \in C(\mathrm{sp}(a))$, we use the shorthand $f := \gamma^{-1}(f)$ to reuse the same function symbols for the sake of sanity. Indeed, this is sensible due to the following argument. If $p \in \mathbb{C}[z, \overline{z}]$ is a polynomial, then using $p(a, a^*) \in A_0$, by definition of A_0. Using Stone-Weierstraß for complex numbers, polynomials of this form are dense in $C(\mathrm{sp}(a))$, so for any $f \in C(\mathrm{sp}(a))$, we can define $f(a) := \gamma(f)$

As a corollary, we also obtain the spectral mapping theorem.

Theorem 3.5. *If $a \in A$ is normal and $f \in C(\mathrm{sp}(a))$, then $f(a) \in C^*(a, 1_A) \subseteq A$ is normal and $\mathrm{sp}(f(a)) = f(\mathrm{sp}(a))$.*

References

[Str20] Karen R. Strung. *An Introduction to C*-Algebras and the Classification Program.* Advanced Courses in Mathematics - CRM Barcelona. Birkhäuser Cham, 2020.